FILE

ABBOTSLEIGH

AUGUST 2006

YEAR 12

ASSESSMENT 4

TRIAL HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes.
- Working time – 2 hours.
- Write using blue or black pen.
- Board-approved calculators may be used
- A table of standard integrals is provided with this paper
- All necessary working should be shown in every question

Total marks - 84

- Attempt Questions 1-7
- All questions are of equal value
- Answer each question in a separate writing booklet

Total marks – 84
Attempt Questions 1–7
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra booklets are available.

Question 1 (12 marks) Use a SEPARATE writing booklet. — Marks

(a) Differentiate $y=\sin^{-1}(x^2)$. — 2

(b) Use the substitution $u=4-x$ to evaluate $\int_0^3 \frac{x}{\sqrt{4-x}}dx$. — 3

(c) Evaluate $\int_1^{\sqrt{2}} \frac{dx}{\sqrt{2-x^2}}$. — 2

(d) Find $\int \sin^2 3x\,dx$. — 2

(e) Solve $\frac{1}{x+2} \leq 2$ — 3

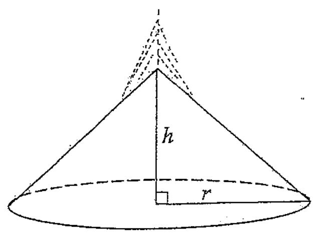
Question 2 (12 marks) Use a SEPARATE writing booklet. Marks

(a) Solve the equation $\sin 2\theta = 2\cos^2\theta$ for $0 \le \theta \le 2\pi$. 3

(b) (i) Express $\sin x + \cos x$ in the form $R\sin(x+\alpha)$. 2

(ii) Hence, or otherwise, sketch the graph of $y = \sin x + \cos x$, showing the endpoints, in the domain $0 \le \theta \le 2\pi$. 2

(c) Wheat runs from a hole in a silo at a constant rate and forms a conical heap whose base radius, r is three times its height, h.



NOT TO SCALE

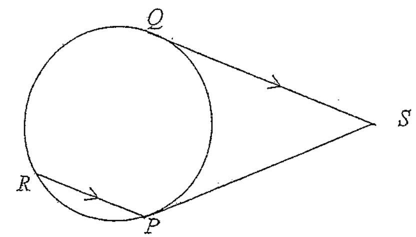
After 1 minute, the height of the heap is 20 cm. Find:

(i) The volume of the conical heap after 1 minute. Leave your answer in terms of π. 2

(ii) The rate at which the height of the heap is rising after 1 minute. 3

Question 3 (12 marks) Use a SEPARATE writing booklet. Marks

(a) 3



NOT TO SCALE

P and Q are points on a circle and the tangents to the circle at P and Q meet at S.
R is a point on the circle so that RP is parallel to QS.

Copy or trace the diagram into your writing booklet.

Prove that $QP = QR$.

(b) (i) Find the value of k if $2x+1$ is a factor of $P(x) = 2x^3 - x^2 + kx + 1$. 2

(ii) Show that when k has this value, $P(x)$ has only the one real root, $x = -\frac{1}{2}$. 3

(c) The rate at which a body cools in air is proportional to the difference between its temperature, T and the constant temperature, P of the surrounding air. This rate can be expressed by

$$\frac{dT}{dt} = k(T - P)$$

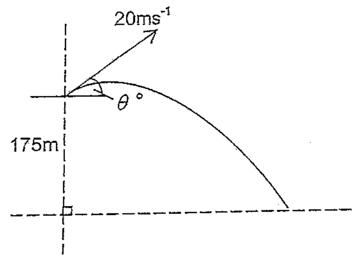
(i) Show that $T = P + Ae^{kt}$ where A and k are constants, satisfies this equation. 1

(ii) A heated body cools from 40° to 30° in 1 hour. The air temperature around the body is 20° C. Find the temperature of the body after a further 2 hours. 3

Question 4 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) When $(3+x)^n$ is written as a polynomial in x, the coefficient of x^4 is twice the coefficient of x^3. Find the value of n. **4**

(b) A man standing on top of a vertical cliff throws a stone into the air at an angle θ to the horizontal. The top of the vertical cliff is 175 metres above a flat sea.



NOT TO SCALE

The angle of projection to the horizontal is θ , as shown.
Let (x, y) be the position of the stone at time t seconds after being thrown.
The initial velocity of the stone is 20 ms^{-1}.

You may assume that the path of the stone where the acceleration due to gravity is -10 ms^{-2}, is given by the parametric equations below

$$x = 20t\cos\theta$$
$$y = 20t\sin\theta - 5t^2 + 175$$

(Do NOT prove these equations.)

The angle of projection of the bullet to the horizontal, θ is 30°.

(i) Find the time it takes for the stone to hit the water. **3**

(ii) Find the speed at which the stone hits the water. **2**

(c) A particle moves on a line so that its distance from the origin at time t seconds is x cm. Its acceleration is given by

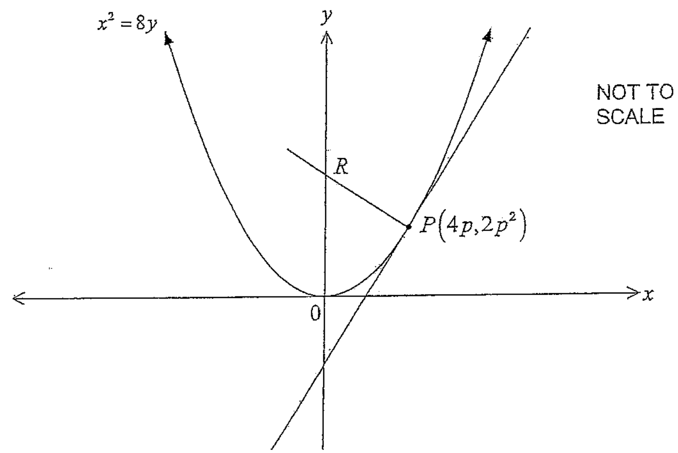
$$\frac{d^2x}{dt^2} = 10x - 2x^3$$

(i) If its velocity is v and the particle changes direction 1 cm to the right of the origin, find v^2 in terms of x. **2**

(ii) Explain why the particle can never reach the origin. **1**

Question 5 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a)



NOT TO SCALE

Consider the variable point $P(4p, 2p^2)$ on the parabola $x^2 = 8y$.

(i) Prove that the equation of the normal at P is $x + py = 2p^3 + 4p$. **2**

(ii) Find the coordinates of the point R, where the normal at P intersects the y-axis. **1**

(iii) If M of the midpoint of PR, find the equation of the locus of M in Cartesian form. **2**

(b) Use mathematical induction to prove that for all integers $n = 1, 2, 3, \ldots$ **3**

$$1\times1! + 2\times2! + 3\times3! + \ldots\ldots n\times n! = (n+1)! - 1$$

(c) The three roots of the polynomial equation $x^3 - 3px^2 + 2qx - r = 0$ form an arithmetic series.

Prove that $r = 2p(q - p^2)$. **4**

Question 6 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Sketch the graph of the function $f(x)=x^3-1$ in the domain $0 \le x \le 2$, clearly showing the coordinates of any points of intersection with the axes. 1

(ii) On the same diagram sketch the graph of the inverse function $f^{-1}(x)$, showing the coordinates of any points of intersection with the axes. 2

(iii) Explain why the $x-$coordinate of any point of intersection of the graphs $y=f(x)$ and $y=f^{-1}(x)$ satisfies the equation $x^3-x-1=0$. 1

(iv) Show that the equation $x^3-x-1=0$ has a root between $x=1$ and $x=2$. 2

(v) Taking $x=1.5$ as the first approximation to the root, use one application of Newton's method to find a better approximation to the root, correct to 3 significant figures. 3

(b) The coefficient of x^k in $(1+x)^n$ where n is a positive integer is denoted as $\binom{n}{k}$.

Prove that $\binom{n}{1}+2\binom{n}{2}+3\binom{n}{3}+...(n-1)\binom{n}{n-1}=n(2^{n-1}-1)$ 3

Question 7 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Solve the equation $(n+\frac{1}{n})^2-5(n+\frac{1}{n})+6=0$. 2

(b) (i) Show that the function $y=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$ has no stationary points. 3

(ii) Prove that the lines $y=\pm 1$ are asymptotes. 2

(iii) Sketch the curve. 1

(iv) If k is a positive constant, show that the area in the first quadrant enclosed by the above curve and the lines $y=1$, $x=0$ and $x=k$ is given by

Area = $k-\ln(e^k+e^{-k})+\ln 2$ 2

(v) Prove that for all positive values of k, this area is always less than $\log_e 2$. 2

END OF PAPER

FILE

2006 EXTENSION I TRIAL SOLUTIONS

QUESTION 1

a) $\frac{dy}{dx} = \frac{2x}{\sqrt{1-(x^2)^2}}$

$= \frac{2x}{\sqrt{1-x^4}}$

b) Let $u = 4 - x$ $\therefore du = -dx$

$x = 3,\ u = 1$

$x = 0,\ u = 4$

$\therefore I = \int_4^1 \frac{4-u}{\sqrt{u}} \times -du$

$= \int_1^4 \frac{4-u}{\sqrt{u}}\, du$

$= \int_1^4 \left(4u^{-1/2} - u^{1/2}\right) du$

$= \left[2\times 4u^{1/2} - \frac{2}{3}u^{3/2}\right]_1^4$

$= 8\times\sqrt{4} - \frac{2}{3}\times 4\times\sqrt{4} - \left(8 - \frac{2}{3}\right)$

$= 16 - \frac{16}{3} - 8 + \frac{2}{3} = 3\frac{1}{3}$

c) $\int_1^{\sqrt{2}} \frac{dx}{\sqrt{(\sqrt{2})^2 - x^2}}$

$= \left[\sin^{-1}\frac{x}{\sqrt{2}}\right]_1^{\sqrt{2}}$

$= \sin^{-1}\frac{\sqrt{2}}{\sqrt{2}} - \sin^{-1}\frac{1}{\sqrt{2}}$

$= \frac{\pi}{2} - \frac{\pi}{4}$

$= \frac{\pi}{4}$

d) $I = \int \frac{1}{2}(1 - \cos 6x)\,dx$ $\quad \cos 2x = 1 - 2\sin^2 x$ $\quad \therefore \sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$= \frac{1}{2}\left[x - \frac{\sin 6x}{6}\right] + C$

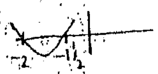
$= \frac{x}{2} - \frac{\sin 6x}{12} + C$

e) $\frac{1}{(x+2)} \le 2$ $\quad x \ne -2,\ \times$ b.s. by $(x+2)^2$

$(x+2) \le 2(x+2)^2$

$(x+2)(2(x+2) - 1) \ge 0$

$(x+2)(2x+3) \ge 0$



$x < -2$ or $x \ge -1\frac{1}{2}$

QUESTION 2

a) $\sin 2\theta = 2\cos^2\theta$

$2\sin\theta\cos\theta = 2\cos^2\theta$

$\cos\theta(\sin\theta - \cos\theta) = 0$

$\cos\theta = 0$ or $\tan\theta = 1$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{4}$ or $\frac{5\pi}{4}$

b) Let $\sin x + \cos x = R\sin(x+\alpha)$

(i) $\sin x + \cos x = R\sin x\cos\alpha + R\cos x\sin\alpha$

$\therefore R\cos\alpha = 1$ ①

$R\sin\alpha = 1$ ②

② ÷ ① $\tan\alpha = 1$

$\alpha = \frac{\pi}{4}$

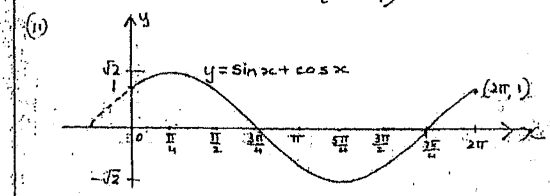
$①^2 + ②^2$ $\quad R^2\cos^2\alpha + R^2\sin^2\alpha = 1 + 1$

$R^2 = 2$

$R = \sqrt{2}$

$\therefore \sin x + \cos x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$

(ii)



c) $V = \frac{1}{3}\pi r^2 h$ $\quad r = 3h$

(i) $\therefore V = \frac{1}{3}\pi(3h)^2 \times h$

$= 3\pi h^3$

at $t = 1$, $h = 20$ $\quad \therefore V = 3\pi(20)^3$

$= 24000\pi$ cm^3

(ii) $\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

$\frac{dV}{dh} = 9\pi h^2$ at $t = 1$, $h = 20$

$\therefore \frac{dV}{dh} = 9\pi(20)^2$

$= 3600\pi$

$\frac{dV}{dt} = 24000\pi$ (from (i))

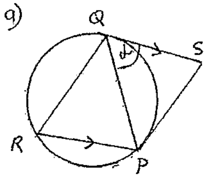
$\therefore \frac{dh}{dt} = \frac{1}{3600\pi} \times 24000\pi$

$= \frac{20}{3}$

$\therefore$ height is rising at $\frac{20}{3}$ cm / minute

QUESTION 3

a)



Join QP, QR

Let $\angle SQP = \alpha$

$\angle QRP = \angle PQS = \alpha$ ($\angle$ between a tangent and a chord = $\angle$ in alt. segment)

$\angle RPQ = \angle SQP = \alpha$ (alt. $\angle^s$, $RP \parallel QS$)

$\therefore \angle QRP = \angle QPR = \alpha$

$\therefore \Delta QRP$ is isosceles

$\therefore QR = QP$ (sides opp. $= \angle^s$ in isosc. Δ)

b) (i) If $2x+1$ is a factor, $P(-\frac{1}{2}) = 0$

$2(-\frac{1}{2})^3 - (-\frac{1}{2})^2 + k \times -\frac{1}{2} + 1 = 0$

$-\frac{1}{4} - \frac{1}{4} - \frac{k}{2} + 1 = 0$

$\therefore k = 1$

(ii) $\therefore P(x) = (2x+1)\cdot Q(x)$

$= (2x+1)(x^2 - x + 1)$

For $x^2 - x + 1$ to have a root, $\Delta \ge 0$

$\Delta = (-1)^2 - 4\times1\times1$

$= -3$

< 0 $\quad \therefore x^2 - x + 1$ has no roots

So $x = -\frac{1}{2}$ is the only root.

c) (i) If $T = P + Ae^{kt}$

$\frac{dT}{dt} = A \times ke^{kt}$

$= k \times Ae^{kt}$

$= k \times (T - P)$

$\therefore T = P + Ae^{kt}$

c) continued

(ii) $P = 20$, $\quad t = 0$, $T = 40$ + $t = 1$, $T = 30$

$T = 20 + Ae^{kt}$

$40 = 20 + Ae^0 \Rightarrow A = 20$

$\therefore T = 20 + 20e^{kt}$

$30 = 20 + 20e^k$

$e^k = \frac{10}{20} = 0.5$

$k = \ln 0.5$

$\therefore T = 20 + 20e^{\ln 0.5 \times t}$

$t = 3$, $\quad T = 20 + 20e^{\ln 0.5 \times 3}$

$= 22.5^\circ$

$\therefore$ temperature is 22.5° after 3 hours

QUESTION 4

a) T_5 has x^4 term

coeff of $x^4 = {}^nC_4 \; 3^{n-4} \times 1^4$

coeff of $x^3 = {}^nC_3 \; 3^{n-3} \times 1^3$

coeff of $x^4 = 2 \times$ coeff of x^3

$\therefore \dfrac{n(n-1)(n-2)(n-3)}{4\times3\times2\times1} \times 3^{n-4}$

$= \dfrac{2\times n(n-1)(n-2)}{3\times2\times1} \times 3^{n-3}$

$\therefore \dfrac{(n-3)}{4} \times 3^{-1} = 2$

$\dfrac{(n-3)}{4} \times \dfrac{1}{3} = 2$

$n-3 = 24$

$n = 27$

b) $x = 20t \times \cos 30^\circ$

$x = 20t \times \dfrac{\sqrt{3}}{2}$

$= 10\sqrt{3}\,t$

$\dot{x} = 10\sqrt{3}$

$y = 20t \times \sin 30^\circ - 5t^2 + 175$

$y = 10t - 5t^2 + 175$

(i) stone hits water at $y = 0$

$5t^2 - 10t - 175 = 0$

$t^2 - 2t - 35 = 0$

$(t-7)(t+5) = 0$

$t-7=0$ or $t+5=0$

$\therefore t = 7$ or -5

but $t \geq 0 \quad \therefore t = 7$

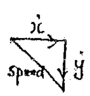
$\therefore$ It takes 7 seconds to hit the water.

(ii) at $t = 7$, $\dot{x} = 10\sqrt{3}$

$\dot{y} = 10 - 10t$

$\therefore \dot{y} = 10 - 10 \times 7$

$= -60$



speed $= \sqrt{(10\sqrt{3})^2 + (60)^2}$

$\doteqdot 62.4$ m/s

c) $a = \dfrac{d}{dx}\left(\tfrac{1}{2}v^2\right) = 10x - 2x^3$

(i) $\tfrac{1}{2}v^2 = 5x^2 - \dfrac{x^4}{2} + C$

$v = 0, \; x = 1$

$\therefore 0 = 5 - \tfrac{1}{2} + C$

$\therefore \tfrac{1}{2}v^2 = 5x^2 - \dfrac{x^4}{2} - 4\tfrac{1}{2}$

$v^2 = 10x^2 - x^4 - 9$

(ii) For the particle to reach the origin, $x = 0$

$\therefore v^2 = -9$ which is impossible

$\therefore$ The particle can never reach the origin.

QUESTION 5

a) (i) $x^2 = 8y \Rightarrow y = \dfrac{x^2}{8}$

$\dfrac{dy}{dx} = \dfrac{2x}{8}$

$= \dfrac{x}{4}$

at $P(4p, 2p^2)$, m of tangent $= \dfrac{4p}{4}$

$= p$

$\therefore$ m of normal is $-\dfrac{1}{p}$ since $m_1 m_2 = -1$ for perpendicular lines.

eq'n of normal: $y - 2p^2 = -\dfrac{1}{p}(x - 4p)$

$py - 2p^3 = -x + 4p$

$\therefore x + py = 2p^3 + 4p$ is the eq'n of the normal.

(ii) at R, $x = 0$

$\therefore 0 + py = 2p^3 + 4p$

$y = 2p^2 + 4$

$\therefore R(0, 2p^2 + 4)$

(iii) $M\left(\dfrac{4p+0}{2}, \dfrac{2p^2 + 2p^2 + 4}{2}\right)$

$\therefore M(2p, 2p^2 + 2)$

$x = 2p$ into $y = 2p^2 + 2$

$y = 2\left(\dfrac{x}{2}\right)^2 + 2$

$\therefore$ eq'n of locus is $y = \dfrac{x^2}{2} + 2$

b) Show that the result is true for $n = 1$

LHS $= 1\times1!$

$= 1$

RHS $= (1+1)! - 1$

$= 2 - 1$

$= 1$

$=$ LHS $\quad \therefore$ result is true for $n = 1$

Assume that the result is true for $n = k$

ie assume $1\times1! + 2\times2! + \dots + k\times k! = (k+1)! - 1$

Using this assumption, show that the result is true for $n = k+1$.

ie show $1\times1! + 2\times2! + \dots + k\times k! + (k+1)(k+1)!$

$= (k+2)! - 1$

is true

LHS $= (k+1)! - 1 + (k+1)(k+1)!$

$= (k+1)!\,(1 + k + 1) - 1$

$= (k+1)!\,(k+2) - 1$

$= (k+2)! - 1$

$=$ RHS

$\therefore$ if the result is true for $n = k$ then it is also true for $n = k+1$

By the process of mathematical induction, this result is true for all integers $n = 1, 2, 3, \dots$

c) $P(x) = x^3 - 3px^2 + 2qx - r = 0$

let roots be $\alpha - d, \alpha, \alpha + d$.

sum of roots: $\alpha - d + \alpha + \alpha + d = 3p$

$3\alpha = 3p$

$\alpha = p \qquad (1)$

sum of product of roots, 2 at a time:

$\alpha(\alpha - d) + \alpha(\alpha + d) + (\alpha - d)(\alpha + d) = 2q$

$\alpha^2 - \alpha d + \alpha^2 + \alpha d + \alpha^2 - d^2 = 2q$

$3\alpha^2 - d^2 = 2q$

but $\alpha = p \quad \therefore 3p^2 - d^2 = 2q$

$d^2 = 3p^2 - 2q$

product of roots: $(\alpha - d)\cdot\alpha\cdot(\alpha + d) = r$

$\alpha(\alpha^2 - d^2) = r$

$\alpha^3 - \alpha d^2 = r$

$p^3 - p\times(3p^2 - 2q) = r$

$\therefore r = p^3 - 3p^3 + 2pq$

$= -2p^3 + 2pq$

$r = 2p(q - p^2)$

or from (1) $\alpha = p$ is a soln so

$P(\alpha) = 0$ and $P(p) = 0$

so $p^3 - 3p\,p^2 + 2q\,p - r = 0$

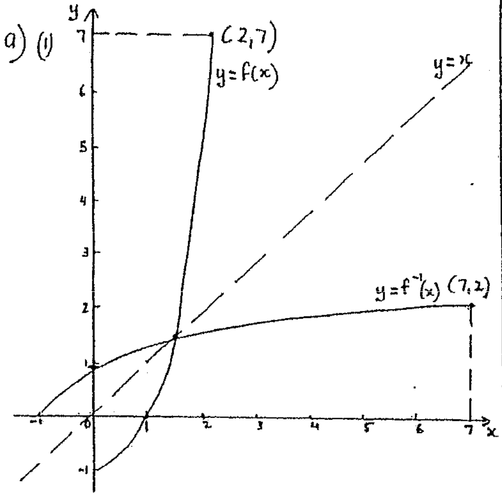
$-2p^3 + 2qp - r = 0$

$r = 2qp - 2p^3$

$r = 2p(q - p^2)$

QUESTION 6

a) (i)



(ii) Any function and its inverse must intersect in the line $y=x$ since this is their axis of symmetry.

pt of intersection of $y=x^3-1$ and $y=x$

$x^3-1=x \quad \therefore x^3-x-1=0$

(iv) If $x=1$, $g(x)=1-1-1$
$= -1$
< 0

If $x=2$, $g(x)=8-2-1$
$=5$
>0

$g(x)$ changes sign $\therefore$ there is a root between $x=1$ and $x=-1$

(v) $g'(x)=3x^2-1$

let $x_1=1\cdot5$

$$x_2 = x_1 - \frac{g(x_1)}{g'(x_1)}$$

$$= 1\cdot5 - \frac{(1\cdot5)^3-1\cdot5-1}{3(1\cdot5)^2-1}$$

$= 1\cdot3478\ldots$

$\doteqdot 1\cdot35$

$\therefore$ a better approximation to the root is $x=1\cdot35$

b) $(1+x)^n = 1+{}^nC_1x+{}^nC_2x^2+\ldots{}^nC_{n-1}x^{n-1}+{}^nC_nx^n$

Differentiating both sides with respect to x:

$n(1+x)^{n-1} = {}^nC_1+2\,{}^nC_2x+3\,{}^nC_3x^2+\ldots(n-1)\cdot{}^nC_{n-1}x^{n-2}+nx^{n-1}$

Let $x=1$,

$n\cdot2^{n-1} = {}^nC_1+2\,{}^nC_2+3\,{}^nC_3+\ldots\ldots+(n-1)\,{}^nC_{n-1}+n$

${}^nC_1+2\,{}^nC_2+3\,{}^nC_3+\ldots\ldots+(n-1)\,{}^nC_{n-1} = n\cdot2^{n-1}-n$

$\therefore \binom{n}{1}+2\binom{n}{2}+3\binom{n}{3}+\ldots\ldots+(n-1)\binom{n}{n-1} = n(2^{n-1}-1)$

QUESTION 7

a) let $m = n+\frac{1}{n}$

$\therefore m^2-5m+6=0$

$(m-3)(m-2)=0$

$m=3$ or 2

$\therefore n+\frac{1}{n}=3$ or $n+\frac{1}{n}=2$

$n^2-3n+1=0$ or $n^2-2n+1=0$

$n=\frac{3\pm\sqrt{9-4}}{2}$ $\qquad (n-1)^2=0$

$\therefore n=\frac{3\pm\sqrt5}{2}$ or 1

b) (i) $$\frac{dy}{dx} = \frac{(e^x+e^{-x})(e^x+e^{-x})-(e^x-e^{-x})(e^x-e^{-x})}{(e^x+e^{-x})^2}$$

$$= \frac{e^{2x}+2+e^{-2x}-e^{-2x}+2-e^{-2x}}{(e^x+e^{-x})^2}$$

$$= \frac{4}{(e^x+e^{-x})^2}$$

$\neq 0$ for any value of x

$\therefore$ The function has no stationary points.

(ii) If $y=1$, then $\frac{e^x-e^{-x}}{e^x+e^{-x}}=1$

$e^x-e^{-x}=e^x+e^{-x}$

$\therefore 2e^{-x}=0$

$e^{-x}=0$ which is not possible $\therefore y\neq1$

If $y=-1$ then $\frac{e^x-e^{-x}}{e^x+e^{-x}}=-1$

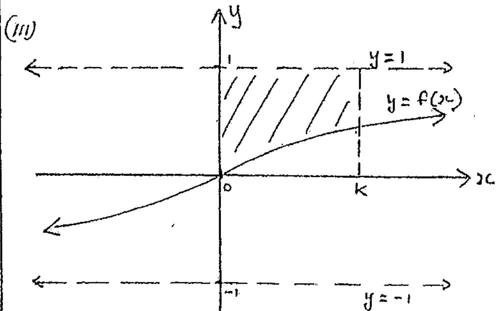
$e^x-e^{-x}=-e^x-e^{-x}$

$2e^x=0$

$\therefore e^x=0$ which is impossible.

$\therefore y=\pm1$ are asymptotes

(iii)



(iv) $$\text{Area} = k-\int_0^k \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx$$

$= k-\left[\ln(e^x+e^{-x})\right]_0^k$

$= k-\ln(e^k+e^{-k})+\ln(e^0+e^0)$

$= k-\ln(e^k+e^{-k})+\ln2$

(v) Area $= k\times\ln e-\ln(e^k+e^{-k})+\ln2$

$= \ln e^k-\ln(e^k+e^{-k})+\ln2$

$= \ln\frac{e^k}{e^k+e^{-k}}+\ln2$

since e^k and e^{-k} are both >0 for all k,

$\frac{e^k}{e^k+e^{-k}}<1$ for all k

$\therefore \ln\frac{e^k}{e^k+e^{-k}}<0$

$\therefore$ Area $<\ln2$ for all values of k